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On Neumann's Method for the Exterior Neumann Problem for the Helmholtz Equation

RALPH E. KLEINMAN*

Department of Mathematics, University of Delaware, Newark, Delaware 19711

AND

WOLFGANG L. WENDLAND†

Fachbereich Mathematik, Technische Hochschule Darmstadt, 61 Darmstadt, West Germany

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INTRODUCTION

Integral equation methods have long been instrumental in proving existence and uniqueness of solutions of scattering problems (see Dolph [16] and Hönl, Maue, and Westphal [26] for extensive references), but only recently has Neumann's classical iteration method been adapted to actually construct solutions. Ahner and Kleinman [3] used this method to solve the three-dimensional exterior Neumann problem in time harmonic acoustic scattering theory for low frequencies. The desired solution was represented by the single and double boundary layer potentials obtained from Green's formula which led to an integral equation for the unknown boundary values of the solution. This integral equation was regularized in the sense that the unknown function appeared in such a way as to vanish at the singularity of the kernel. The proof of convergence of the Neumann series in [3] was based on the contraction property of the operator with respect to the maximum norm. Unfortunately, this norm inequality is not true; [3, formula (4.35)] does not follow from [3, (4.34)]. But if we replace the maximum norm $\| u \|_0$ by the equivalent norm $| u | = \max | u(p_1) - u(p_2)| + \alpha \| u \|_0$ with a suitable small constant $\alpha > 0$, then, according to C. Neumann's classical proof (see [19, pp. 203ff.]) and its completion by Schober [52] it follows that for *convex*

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† Research begun while this author was a Visiting Unidel Chair Professor at the University of Delaware.

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regions and small wavenumbers k the desired contraction property holds with respect to $|\cdot|$ (Lemma A in Appendix A). Hence, Neumann's method converges in this case and the conclusions in [3] remain valid if one uses this norm. But this method is still limited to *convex* scattering bodies. In the harmonic limit ($k = 0$) another treatment was given by Wendland [58] in which the convergence of the Neumann series was proven for more general bodies. The proof was based on Plemelj's theorem on the eigenvalues of the integral equation (see [27, p. 134]) and an analytic transformation of the integral operators which can be found in [33, pp. 118ff.]. Kress and Roach [36] investigated the Neumann series involving more general fundamental solutions by the use of extremal properties of the eigenvalues. They indicated that modifications of the classical series converge even faster. In the present paper we extend the approach [58] to the Helmholtz equation for small wavenumbers and the general class of boundary surfaces characterized by Burago, Mazja, and Sapozhnikova [10, 11], and by Kral [59]. This class includes piecewise Lyapunoff surfaces excluding spines [58]. We show that the spectral radius of the operator corresponding to Neumann's method becomes less than 1 (independent of the convexity) provided the wavenumber is small enough. Hence, Neumann's series converges.

This convergence proof is valid for any function space in which the spectrum is unchanged. Therefore, at least for smooth enough surfaces Γ, the Neumann series converges in all function spaces $L^p(\Gamma)$ with $1 \leq p \leq \infty$ as well as for C^0.

For the numerical computations the equations are discretized using suitable quadrature formulas. Because this gives rise to large systems of linear equations, it is essential that the successive approximation method for the original equation converges for the numerical equations as well. Using the approximation theory for integral equations as developed by Anselone [4], we show that the numerical equations can be solved approximately and that the convergence rate does not depend on the number of the discrete equations. If the surface Γ has corners and edges then the same result follows from a modification given by [8] which was applied for $k = 0$ in [58].

The investigations reported here which establish numerical convergence of the iterates and take into account the influence of corners generalize to nonzero k these results of Wendland. Some numerical computations based on this method will be presented in a forthcoming paper by Jones and Kleinman [32].

Numerical computations for three-dimensional scattering problems based on alternate integral equation formulations have been presented by Brundrit [9], Chertok [14], Copley [15], Chen and Schweikert [13], Fenlon [17], Hess [24], Peterson and Ström [46], and, for $k = 0$, Hess and Smith [25], Lynn and Tinlake [60], Wendland [58]. Comparison of the efficiency of the various methods are not yet available but we expect that the iteration method discussed in this paper will prove numerically superior, at least for some range of wave number.

As is evident in the following, the regularized integral equation does not have a unique solution at eigenvalues of the interior Dirichlet problem, a complication shared by the alternate formulations mentioned above. Supplementary conditions derived from the Helmholtz representation which guarantee a unique solution even at interior eigenvalues are found in Burton and Miller [12] and Kleinman and Roach [35], although these conditions severely complicate numerical computations. Another method of removing the difficulty is based on an integral equation derived from an assumption that the unknown field may be represented as a combination of simple and double layer potentials with the same unknown density [7, 41, 42, 45]. Computations based on this equation are available in two-dimensional problems, Greenspan and Werner [20], Kussmaul and Werner [39], Kussmaul [38], and Bolomey and Tabbara [5] (who designate the technique as the "hybrid potential method"); however, as yet we are unaware of applications of this method to the three-dimensional problem. A different approach to resolving the indeterminacy at interior eigenvalues, based on the fact that the field vanishes interior to the surface, was proposed by Kupradze (see [37] for description and earlier references), Schenk [51] and Waterman [56], whose analysis has given rise to the term "extended boundary condition" to characterize this approach. Weck extended the method in [57]. Bolomey and Tabbara [5] conclude from their two-dimensional computations that the hybrid potential method is more reliable than that based on the extended boundary condition for computing fields near interior eigenvalues. Ursell [55] derived an integral equation, using the Green's function for a surface interior to the surface of interest rather than the free space Green's function, which has a unique solution for all positive wavenumbers; however, no computations based on this equation have been carried out. Jones [31] has presented another integral equation which in some sense combines features of the hybrid potential and extended boundary condition methods to guarantee unique solutions, but again no computations are available yet. Other numerical approaches, primarily directed to electromagnetic scattering but with relevance to the acoustic problem, are described in Poggio and Miller [47].

The present methods may be extended to integral equations arising in other problems if the analog of Plemelj's theorem is available. This is the case for the equations of elasticity where Ahner and Hsiao [1, 2] have established convergence of the Neumann series. Extensions to transition problems for the Helmholtz equation have been carried out by Kittappa and Kleinman [34]. It should also be possible to extend the method to the exterior Dirichlet problem with the help of one of the integral equation systems found in [35], but this is yet to be done.

1. The Integral Equation

We are concerned with a constructive integral equation method for the exterior Neumann problem:

$$\Delta u + k^2 u = 0 \qquad \text{in} \qquad R^3 - \bar{G}, \quad \operatorname{Im} k \geq 0, \tag{1.1}$$

$$\partial u/\partial n = g \qquad \text{on} \qquad \partial G = \Gamma \tag{1.2}$$

and

$$\lim_{r \to \infty} r(\partial/\partial r - ik)\, u = 0. \tag{1.3}$$

The problem is not completely formulated without additional assumptions on the domain G and the desired solution. It is well known [49, 6], in two-dimensional problems for boundaries with edges, additional conditions are needed to guarantee a unique solution (see [22] for an interesting modern treatment of two dimensional problems). These "edge conditions" may take the form of a requirement of locally finite energy [43], or, equivalently, a prescribed order condition on the dependence of the solution as a function of distance from the edge [23; 28–30]. The order conditions have been developed primarily for electromagnetic scattering from discs and apertures in plane screens; however, the finite energy condition is equally applicable in acoustic scattering for boundaries enclosing nonzero volume where it takes the form [26]

$$\int_D \{|u|^2 + |\nabla u|^2\}\, d\tau < \infty \tag{1.4}$$

for every bounded region $D \subset (R^3 - G)$.

Observe that D may include ∂G or portions thereof. Rather than impose this condition, however, we shall show that it is fulfilled as a consequence of the assumption on the domain and alternative requirements on the solution u. The assumptions on the domain are those of [10] such that Green's theorem, the Helmholtz representation formula, and the integral equation method are available. We assume that $\partial G = \Gamma$ is a closed bounded surface with finite Lebesgue area and zero volume. Let us denote by $\mathbf{n}(p)$ the outer normal of Γ which is defined almost everywhere. $\partial/\partial n_p$ denotes the normal derivative. Then

$$d\Omega_q(p) = -\partial/\partial n_p\,(1/|p - q|)\, dS_p = (\mathbf{n}(p) \cdot (\mathbf{p} - \mathbf{q})/|p - q|^3)\, dS_p \tag{1.5}$$

is well defined for almost all $p \in \Gamma$, $p \neq q$.

In [10] it was shown that under the assumption

$$\sup_{q \in R^3 - \Gamma} \int_\Gamma | d\Omega_q(p)| < \infty, \tag{1.6}$$

the potential theory of simple and double layer potentials can be developed, and it was also proven that the inequality (1.6) holds even for $q \in \Gamma$.

Because we wish to use the integral equation method we restrict corners and edges by the additional condition

$$\lim_{\delta \to 0}(\sup_{q \in \Gamma} W_\delta(q)) = \omega < 1, \tag{1.7}$$

where

$$W_\delta(q) = (1/2\pi) \left\{ \int_{0<|p-q|<\delta,\, p \in \Gamma} | d\Omega_q(p)| + | 2\pi - \Omega(q)| \right\} \tag{1.8}$$

and

$$\Omega(q) = \int_{\Gamma - \{q\}} d\Omega_q(p) \tag{1.9}$$

(see also [58]).

Equation (1.7) implies that $2\pi(1 - \omega) \leq \Omega(q) \leq 2\pi(1 + \omega)$ for all $q \in \Gamma$ and this constitutes a cone condition [with vertex angle $2 \cos^{-1} \omega$] on both sides of Γ. (Note that on smooth points of Γ, $\Omega(q) = 2\pi$ and $\omega = 0$.) This guarantees that Fredholm's theorem is applicable for the integral equation on Γ. The desired solution u is required to belong to $C^0(R^3 - G) \cap C^2(R^3 - \bar{G})$, to fulfill the radiation condition (1.3) uniformly with respect to the angle, and to have a normal derivative in the sense of a boundary flow defined as follows (see [11, p. 20]). Let Γ_m be a sequence of smooth surfaces each of which is given by a continuous mapping ϕ_m of Γ in R^3. Let us assume that $\phi_m(p) \to p \in \Gamma$ uniformly on Γ ($m \to \infty$) and $\int dS_m \to \int dS$. We say that u has *exterior boundary flow* $\partial u/\partial n$ if for any function $\phi \in C^\infty(R^3)$ with compact support and for any sequence $\Gamma_m \subset R^3 - \bar{G}$ converging to Γ in the sense defined above there exists the limit

$$l_u(\phi) = \lim_{m \to \infty} \int_{x \in \Gamma_m} \phi(x)\,(\partial u/\partial n_m)\, dS_m \tag{1.10}$$

defining a *bounded linear functional* on $C^0(\Gamma)$ where $C^0(\Gamma)$ denotes the Banach space of continuous functions on Γ supplied with the maximum norm $\|\cdot\|_0$. Consequently, in (1.2) we mean by $\partial u/\partial n$ and g suitable *distributions* in $C^{0*}(\Gamma)$. As is well known, $g \in C^{0*}(\Gamma)$ can be represented by the Lebesgue–Stieltjes integral

$$\langle g, \phi \rangle = \int_\Gamma \phi \, d\tilde{g}$$

with $\tilde{g}$ being a set function of bounded total variation. The total variation defines the norm on C^{0*} corresponding to $\|\cdot\|_0$ on C^0,

$$\|g\|_{0^*} = \int_\Gamma |d\tilde{g}|.$$

Thus the Neumann condition (1.2) is understood to mean

$$\langle g, \phi\rangle = \int_\Gamma \phi\, d\tilde{g} = \lim_{m\to\infty} \int_{\Gamma_m} \phi(x)\,(\partial u/\partial n_m)\, dS_m\,, \tag{1.11}$$

for all test functions ϕ, where the boundary data g must be of bounded total variation in the sense explained before.

If g has the property that the simple layer potential with density g, i.e.,

$$\langle g, (1/|\cdot - q|)\rangle = \int_\Gamma (1/|p-q|)\, d\tilde{g}(p), \qquad q \notin \Gamma, \tag{1.12}$$

can be *extended* to a continuous function across the surface Γ, then, following [48], we call this class of simple layers C_v, and it includes $L_p(\Gamma)$ for $p > 2$. (This can be shown by a slight generalization of [10, Lemma 13].) To obtain the appropriate integral representation of the solution, let $\{G_m\}$ be the sequence of domains enclosed by $\{\Gamma_m\}$. Then, for $q \in R^3 - \bar{G}_m$, Green's formula yields

$$u(q) = -\frac{1}{4\pi}\int_{\Gamma_m} \frac{e^{ikR}}{R}\frac{\partial u(P)}{\partial n_{p,m}}\, dS_m(p) + \frac{1}{4\pi}\int_{\Gamma_m} u(p)\frac{\partial}{\partial n_{p,m}}\left(\frac{e^{ikR}}{R}\right) dS_m\,, \tag{1.13}$$

where

$$R = |p - q|.$$

The last term can be rewritten with

$$(\partial/\partial n)\, R = -R^2(\partial/\partial n)\,(1/R),$$

so that we have

$$\begin{aligned} u(q) = &-\frac{1}{4\pi}\int_{\Gamma_m} \frac{e^{ikR}}{R}\frac{\partial u(P)}{\partial n_{p,m}}\, dS_m(p) \\ &+ \frac{1}{4\pi}\int_{\Gamma_m} u(p)\, e^{ikR}(1 - ikR)\frac{\partial}{\partial n_{p,m}}\left(\frac{1}{R}\right) dS_m\,. \end{aligned} \tag{1.14}$$

While this formula holds for all smooth surfaces Γ_m, the question of its validity for Γ must be investigated more closely. If m tends to infinity then the first term on the right-hand side converges because we require u to have a boundary flow, and (1.11) holds (q is fixed and $q \notin \Gamma$). The sequence of the second terms in (1.14) converges also as $m \to \infty$, but this is not obvious and follows only because of the important result of [10, Theorem 2]. Therefore, *each solution of the exterior problem* (1.1), (1.2), (1.3) *in the desired class of solutions can be represented by*

$$u(q) = -(1/4\pi) \int_\Gamma (e^{ikR}/R)\, d\tilde{g} - (1/4\pi) \int_\Gamma u(p)\, e^{ikR}(1 - ikR)\, d\Omega_q(p). \quad (1.15)$$

According to Gauss' formula [10, p. 7], this representation is equivalent to the formula

$$u(q) = -(1/4\pi) \int_\Gamma (e^{ikR}/R)\, d\tilde{g} - (1/4\pi) \int_\Gamma (u(p) - u(q))\, d\Omega_q(p)$$
$$- (1/4\pi) \int_\Gamma u(p) \{e^{ikR} - 1 - ikRe^{ikR}\}\, d\Omega_q(p), \qquad q \in R^3 - \bar{G}, \quad (1.16)$$

which was used in [3].

To get an integral equation from this representation, we let q approach $\Gamma = \partial G$. If u belongs to the desired class of solutions, then $u(q)$ on the left-hand side tends continuously to its boundary value. The right-hand side also remains unchanged in form for $q \in \Gamma$ as we now show. The first term on the right-hand side, which for future reference we designate as $f(q')$, can be decomposed into the two limits

$$\lim_{q \to q' \in \Gamma} - (1/4\pi) \int_\Gamma (e^{ikR}/R)\, d\tilde{g}$$
$$= - \lim_{q \to q' \in \Gamma} \left((1/4\pi) \int_\Gamma (1/R)\, d\tilde{g}\right) - (1/4\pi) \int_\Gamma (1/R)(e^{ikR} - 1)\, d\tilde{g} = f(q'). \quad (1.17)$$

The continuity of the second term in (1.17) follows because the boundary flow $\tilde{g}$ has bounded total variation and $(1/R)(e^{ikR} - 1)$ is continuous in p and q. The existence of the limit of the second term in (1.16),

$$\lim_{q \to q' \in \Gamma} - (1/4\pi) \int_\Gamma (u(p) - u(q))\, d\Omega_q(p) = -(1/4\pi) \int_\Gamma (u(p) - u(q'))\, d\Omega_{q'}(p), \quad (1.18)$$

and its continuity follow from (1.6) and the continuity of u.

Similarly, (1.6) implies the existence and continuity of the limit of the third term,

$$\begin{aligned} &\lim_{q \to q' \in \Gamma} - (1/4\pi) \int_\Gamma u(p) \{e^{ikR} - 1 - ikRe^{ikR}\}\, d\Omega_q(p) \\ &\quad = -(1/4\pi) \int_\Gamma u(p) \{e^{ikR} - 1 - ikRe^{ikR}\}\, d\Omega_{q'}(p). \end{aligned} \quad (1.19)$$

Here only the boundedness of u is needed because the remaining expression $\{\cdots\}$ is continuous for all p and q and vanishes at $p = q$ like $O(R^2)$.

The existence and continuity of all these limits imply that the first term in (1.17), viz.,

$$(1/4\pi) \int (1/R)\, d\tilde{g},$$

also has continuous boundary values on Γ. According to [10] this potential can be continuously extended to a harmonic function in the interior. Hence, $g \in C_v$; i.e., *the boundary flow of any solution of the desired class belongs necessarily to* C_v.

Moreover, with our requirements on the solution u, it follows that the "edge condition" (1.4) is satisfied. First of all we observe that

$$u \in C^0(R^3 - G) \Rightarrow \int_D |u|^2\, d\tau < \infty \tag{1.20}$$

for every bounded subdomain $D \subset R^3 - G$. Next we choose a test function $\phi \in C^\infty(R^3)$ with compact support and find

$$\langle g, \phi\rangle = \lim_{m\to\infty} \int_{\Gamma_m} \phi(\partial u/\partial n_m)\, ds_m . \tag{1.21}$$

Let $\Gamma_0 \subset R^3 - \bar{G}$ be a sphere enclosing Γ_m for all m and apply the divergence theorem in the region interior to Γ_0 and exterior to Γ_m, in which case (1.21) becomes

$$\int_\Gamma \phi\, d\tilde{g} = \lim_{m\to\infty} \left[\int_{\Gamma_0} \phi(\partial u/\partial n_0)\, ds - \int_{G_0-G_m} \nabla\cdot(\phi\nabla u)\, d\tau\right]$$

$$= \int_{\Gamma_0} \phi(\partial u/\partial n_0)\, ds + \lim_{m\to\infty} \left[k^2 \int_{G_0-G_m} \phi u\, d\tau - \int_{G_0-G_m} \nabla\phi\cdot\nabla u\, d\tau\right].$$

Since u is continuous, the first limit exists and therefore the last limit exists too. Hence,

$$\int_{G_0-G} \nabla\phi\cdot\nabla u\, d\tau = -\int_\Gamma \phi\, d\tilde{g} + \int_{\Gamma_0} \phi(\partial u/\partial n_0)\, ds + k^2 \int_{G_0-G} \phi u\, d\tau, \tag{1.22}$$

for all $\phi \in C^\infty$. Because the boundary flow g is in C_v there exists a function w harmonic in $R^3 - \bar{G}$ with boundary flow $\partial w/\partial n = g$ on Γ and finite Dirichlet integral according to [11, Theorem 8, 2]. Hence, we have with Green's formula

$$\int_{G_0-G} \nabla\phi\cdot\nabla u\, d\tau = \int_{R^3-G} \nabla\phi\cdot\nabla w + \int_{\Gamma_0} \phi(\partial u/\partial n_0)\, ds + k^2 \int_{G_0-G} \phi u\, d\tau. \tag{1.23}$$

Because of $u \in C^2(R^3 - \bar{G})$, it is now easy to show that the right-hand side of (1.23) is a linear functional on the space of test functions ϕ which is bounded with respect to the Sobolev norm $H^1(G_0 - G)$. This implies that the left-hand side defines a bounded linear functional over $H^1(G_0 - G)$. The Riesz Representation Theorem guarantees that u also belongs to $H^1(G_0 - G)$ which establishes the inequality

$$\int_{G_0-G} |\nabla u|^2 \, d\tau < \infty.$$

This, together with (1.20), shows that the edge condition (1.4) is fulfilled.

In summary, we have shown that the continuous boundary values of any solution u in the desired class satisfy the integral equation

$$u(q) = L_k u + f(q) \qquad \text{on } \Gamma, \tag{1.24}$$

where f is defined in (1.17) and $L_k u$ is defined as

$$\begin{aligned} L_k(q) = & -(1/4\pi) \int_\Gamma [u(p) - u(q)] \, d\Omega_q(p) \\ & -(1/4\pi) \int_\Gamma u(p) \{e^{ikR} - 1 - ikRe^{ikR}\} \, d\Omega_q(p). \end{aligned} \tag{1.25}$$

The operator L_k (1.25) has the properties

$$L_k \colon C^0(\Gamma) \to C^0(\Gamma), \tag{1.26}$$

and

$$\| L_k - L_0 \|_0 \leq (|k|^2/2\pi) \max\{1, e^{-\operatorname{Im} kD}\} \int_\Gamma dS, \tag{1.27}$$

where $D =$ diameter (Γ).

The last inequality follows from [3, formulas (4.37), (4.38), (4.39)] with some elementary computations also under our more general assumptions. Later we shall see that for wavenumbers small enough in absolute value, the integral equation (1.24) is uniquely solvable in $C^0(\Gamma)$. Thus our method excludes critical values of k and has its natural bound at the smallest critical value of k. This is a well-known drawback of the integral equation method if one uses the integral equation (1.24) with (1.25) as we do.

However, our approach has the advantage that (1.24) is solvable by a *successive approximation* which converges for k small in absolute value. If k is not a critical value, i.e., (1.24) has no eigensolutions, then each continuous function u on Γ solving the integral equation (1.24) generates by (1.15) a solution of the exterior problem belonging to the desired class provided $\operatorname{Im} k \geqslant 0$. This can be seen as follows: If u is extended by (1.15) into the exterior then this extension is in $C^0(R^3 - G) \cap C^2(R^3 - \bar{G})$ and satisfies (1.1) and (1.3). It remains to show that u has the given boundary flow g. This is done in Appendix B.

Consequently, the exterior problem (1.1), (1.2), (1.3) *has exactly one solution in the desired class for* $|k|$ *sufficiently small.*

2. Neumann's Method

In this chapter we shall solve the integral equation (1.24) by the successive approximation

$$u^{(n+1)} = L_k u^{(n)} + f, \qquad u^{(0)} = f. \tag{2.1}$$

The convergence of the sequence of iterates follows from the fact that the spectral radius of L_k is less then 1 provided $|k|$ is small enough. This fact is a consequence of Plemelj's theorem and (1.26) as follows.

Remark. Kress and Roach [36] use the iteration

$$u^{(n+1)} = \tfrac{4}{3}L_k u^{(n)} - \tfrac{1}{3}u^{(n)} + \tfrac{4}{3}f \qquad \text{for } k = 0$$

where the operator $\frac{4}{3}L_0 - \frac{1}{3}I$ has a smaller spectral radius than L_0. This can be proven similarly to the following.

We define the operator[1]

$$K_k = I - 2L_k. \tag{2.2}$$

For $k = 0$ the operator K_0 is the integral operator of classical potential theory. For the more general boundary considered here it was investigated in [10; 11 (there called T); 58; 59]. K_0 is continuous from $C^0(\Gamma)$ into $C^0(\Gamma)$ and has the following properties:

Radon's Theorem. *The Fredholm radius of* K_0 *is* $1/\omega$.

ω is defined in (1.7) and the requirement imposed there that $\omega < 1$ implies that the Fredholm radius is greater than 1. Hence, *Fredholm's alternative* holds for

$$I - \lambda K_0 \qquad \text{if} \qquad |\lambda| < 1/\omega \qquad \text{(see [50, p. 191])}$$

Radon proved this theorem in plane domains. In three dimensions, the proof can be found in [11, Theorem 6].

Plemelj's Theorem. *The eigenvalues* λ_ν *of* K_0

$$u = \lambda K_0 u \tag{2.3}$$

[1] K_0 can be written as $K_0 u = (1/2\pi)\{\int_{\Gamma-\{q\}} u(p)\, d\Omega_q(p) + (2\pi - \Omega(q))\, u(q)\}$.

are real for $|\lambda| < 1/\omega$. $\lambda_0 = 1$, *and all the other eigenvalues satisfy* $|\lambda_\nu| > 1$. *Furthermore*, λ_0 *is simple.*

Plemelj proved this Theorem for smooth boundary. With Theorem 8 and Lemma 20 in [10] this proof remains valid for the more general boundaries considered here provided λ is in the Fredholm circle. The eigenvalues of K_0 and L_0 are related in the following way. According to (2.2), μ is an eigenvalue of

$$u = \mu L_0 u \tag{2.4}$$

if and only if

$$\mu = 2\lambda/(\lambda - 1), \tag{2.5}$$

where λ is an eigenvalue of (2.3). Since the Fredholm disk $|\lambda| < 1/\omega$ is mapped by (2.5) into all points in the complex μ-plane such that

$$|(\mu - 2)/\mu| < \omega,$$

i.e., all points exterior to the circle of radius $2\omega/(1 - \omega^2)$ centered at $(2/(1 - \omega^2), 0)$ (see Fig. 1), these points are all Fredholm points of L_0. The point $\mu = 1$ is the image of $\lambda = -1$, therefore it belongs to the resolvent set.

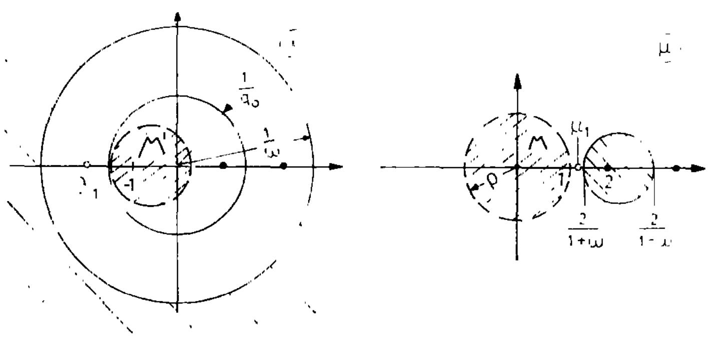

FIG. 1. Fredholm points of K_0 and L_0.

In order to estimate the spectral radius of L_0 we define the number

$$\mu_1 = \min\{2/(\omega + 1), 2\lambda_\nu/(\lambda_\nu - 1)\}, \tag{2.6}$$

where the λ_ν are the eigenvalues of K_0 lying in the disk $|\lambda| < 1/\omega$. Note that $2\lambda_\nu/(\lambda_\nu - 1) > 1$ because of Plemelj's Theorem.

Figure 1 illustrates that eigenvalues λ_ν between -1 and $-1/\omega$ are mapped to the real axis between 1 and $2/(1 + \omega)$. Hence, if there are such eigenvalues, then the one nearest -1 will determine μ_1, otherwise μ_1 equals $2/(1 + \omega)$.

In either case $2 \geq \mu_1 > 1$. For every fixed $\rho < \mu_1$, the points

$$\mu \in M \equiv \{\mu \mid |\mu| \leq \rho < \mu_1\} \tag{2.7}$$

are Fredholm points of L_0. Furthermore, they are not eigenvalues, hence belong to the resolvent set. Consequently, the spectrum $\sigma(L_0)$ is in the complement and

$$\inf_{\mu \in \sigma(L_0)} |\mu| \geq \mu_1 . \tag{2.8}$$

Thus we have the following theorem for the spectral radius of L_0.

THEOREM 2.1. *The spectral radius of L_0 is strictly less than one, i.e.,*

$$\overline{\lim_{j\to\infty}} \| L_0^j \|^{1/j} \leq 1/\mu_1 < 1. \tag{2.9}$$

(See, e.g., [54, pp. 262ff.] where λ has to be replaced by $1/\mu$.)

Remark. (2.9) implies the convergence of Neumann's series for $(I - L_0)^{-1}$ and of the successive approximation (2.1) for $k = 0$. This method for solving Neumann's problem in potential theory for piecewise Lyapunoff surface was established in [58, Satz 8].

The main goal of the present work is the proof that this method of solution remains valid for small but nonzero $|k|$. We show this by establishing that the spectral radius of L_k remains smaller then 1 when $k = 0$ but small enough.

THEOREM 2.2. *For any ρ, $1 < \rho < \mu_1$, there exists a $k_\rho > 0$ such that for all k with $|k| \leq k_\rho$*

$$\overline{\lim_{j\to\infty}} \| L_k^j \|^{1/j} \leq 1/\rho < 1. \tag{2.10}$$

This implies (see [54, p. 262]) the following

COROLLARY. *For all k, $|k| \leq k_\rho$ the Neumann series for $(I - L_k)^{-1}$ converges uniformly as does the successive approximation* (2.1).

Proof of Theorem 2.2. From Theorem 2.1 we know that $(I - \mu L_0)^{-1}$ exists for $|\mu| \leq \rho < \mu_1$. For any $\rho < \mu_1$ we define

$$B_\rho \equiv \operatorname*{Max}_{|\mu| \leq \rho} \|(I - \mu L_0)^{-1}\|_0 . \tag{2.11}$$

From (1.27) it follows that

$$\|(I - \mu L_k) - (I - \mu L_0)\|_0 \leq |\mu| \, |k|^2 C(k) \leq \rho \, |k|^2 C(k), \tag{2.12}$$

where

$$C(k) = (1/2\pi) \max\{1, e^{-\operatorname{Im} k D}\} \cdot \int_\Gamma dS.$$

Now we define k_ρ such that

$$\rho k_\rho^2 C(k_\rho) < B_\rho^{-1} \tag{2.13}$$

holds. Then, for $|k| \leq k_\rho$, (2.11) and (2.12) imply

$$\begin{aligned} \|(I - \mu L_k) - (I - \mu L_0)\|_0 &\leq \rho k_\rho^2 C(k_\rho) \\ &< B_\rho^{-1} \leq \|(I - \mu L_0)^{-1}\|_0^{-1} \end{aligned} \tag{2.14}$$

for all $|\mu| \leq \rho$. By the use of a well-known theorem (see, e.g., [54, p. 164]) this inequality implies the existence of $(I - \mu L_k)^{-1}$ for all these μ, $|\mu| \leq \rho$. That is, all μ, $|\mu| \leq \rho$, are in the resolvent set of L_k, hence the theorem follows.

Now we consider the following chain of Banach spaces:

$$C^0(\Gamma) \subset L^\infty(\Gamma) \subset L^p(\Gamma) \subset L^1(\Gamma) \subset C^{0*}(\Gamma), \qquad 1 < p < \infty. \tag{2.15}$$

If we assume that the boundary surface provides besides (1.6) the *additional estimate*

$$\operatorname*{ess\,sup}_{q \in \Gamma} \int_\Gamma |(\partial/\partial n_q)(1/R)| \, dS(p) < \infty, \tag{2.16}$$

then K_0, and therefore L_0 and L_k, become continuous mappings from each of these spaces into itself [27, inequality (12.23)].

But, unfortunately, the inequality (2.16) is violated even if Γ has only one smooth edge as was shown by Leis [40], (see also [18, 44].) In this case we only know that K_k and L_k are continuous mappings in C^0. Therefore, let us assume for the rest of this chapter that *Γ is a closed Lyapunoff surface.*

In this case, K_0 becomes a weakly singular integral operator. The estimate (2.16) is satisfied and, moreover, *K_0 is compact in each of the spaces* (2.15). This can be seen for $1 < p \leq 2$ by using [53, V, Satz 2, p. 309]. (There we have to use the special constants $n = s = 2$, $\lambda = 1$, $p = q \leq 2$). For $2 \leq p < \infty$ it follows from the compactness of the adjoint operator.

For C^0, the compactness follows in the usual way (e.g., [27, Satz 8.2, p. 107]). The adjoint spaces to those in (2.15) are

$$C^{0*} \supset L^{1*} \supset L^{p'} \supset L^\infty \supset C^{0**}. \tag{2.17}$$

Hence, K_0 is compact there, too. The kernel of $K_k - K_0$ is a continuous function. Thus K_k is also a compact operator in all the spaces (2.15) and (2.17). Then it is easy to show that *the spectra of K_k in all these spaces are the* *same*. (See, e.g., [27, Satz 12.8, p. 193]). Hence, L_k *has the same spectrum in all these spaces too.*

For the Neumann method, considered here, the same location of the eigenvalues implies the following result.

COROLLARY. *If Γ is a closed Lyapunoff surface then Theorem 2.2 holds with each of the operator norms corresponding to each of the spaces in* (2.15) *or* (2.17). *That implies the convergence of Neumann's method in each of these spaces.*

3. THE NUMERICAL EQUATIONS

For the numerical treatment of (1.24) let us assume that Γ is given as a *piecewise Lyapunoff* surface, $\Gamma = \bigcup_{i=1}^{N} S_i$, where S_i is a segment of a Lyapunoff surface. This is a partition of Γ and further subdivision of each S_i into Lyapunoff segments results in a finer partition of Γ, $\Gamma = \bigcup_{j=1}^{M} F_j$, where $M \geqslant N, F_j \cap F_l = \phi$ for $j \neq l$. Denote an arbitrary point in F_j by p_j and let h be a measure of the fineness of the partition:

$$p_j \in F_j, \qquad \max_{i=1,\ldots,M} \operatorname{diam} F_j = h, \qquad |F_j| \equiv \int_{F_j} dS. \tag{3.1}$$

Next we rewrite (1.24) as follows:

$$\begin{aligned} u &= L_k u + f = \tfrac{1}{2}(I - K_k)\, u + f \\ &= \tfrac{1}{2}(I - Q - V)\, u + f, \end{aligned} \tag{3.2}$$

where

$$Qu(q) = u(q)\,((2\pi - \Omega(q))/2\pi) + (1/2\pi) \int_{\Gamma \cap \{p \mid 0 < |p-q| \leqslant \delta\}} u(p)\, d\Omega(p), \tag{3.3}$$

$$\begin{aligned} Vu(q) &= (1/2\pi) \int_{\Gamma \cap \{p \mid |p-q| > \delta\}} u(p)\, d\Omega_q(p) \\ &\quad + (1/2\pi) \int_{\Gamma} u(p) \{e^{ik|p-q|} - 1 - ik\,|p-q|\, e^{ik|p-q|}\} \cdot d\Omega_q(p), \end{aligned} \tag{3.4}$$

and δ is a constant which will be restricted subsequently. We would like to discretize Eq. (3.2) by using an arbitrary quadrature formula having positive weights g_j such that if $f \in C^0(\Gamma)$,

$$\int_{\Gamma} f(p)\, dS_p = \lim_{h \to 0} \sum_j f(p_j)\, g_j\,. \tag{3.5}$$

Whereas for continuous kernels no more assumptions are needed for proving the convergence of the solutions of the discretized equations [4, Chap. 2], even the weak singularity of our kernel on the smooth parts of Γ requires either a special quadrature formula or more restrictions on the partition and the weights. Therefore, we demand the following.

ASSUMPTION. *There is a constant* $c > 0$ *independent of* h *such that the inequalities*

$$g_j/|F_j| \le c, \qquad \textit{and} \qquad c\,|p - p_j| \ge h \qquad \textit{for all } p \in \Gamma - F_j \tag{3.6}$$

hold for all partitions.

The first assumption means that small weights are associated with small F_j. The second assumption implies a restriction on the location of p_j in F_j as well as a restriction on the shape of F_j in order to ensure that p_j is not "too close" to ∂F_j. One way of guaranteeing this is to define p_j to be that point which maximizes the minimum distance to the boundary, i.e.,

$$\max_{p\in F_j}(\min_{q\in\partial F_j} |p - q|) = \min_{q\in\partial F_j} |p_j - q|$$

and then restrict allowable partitions to those for which

$$h \le c \cdot \min_j(\min_{q\in\partial F_j} |p_j - q|).$$

As we shall see, the discretization of the integral operators with such quadrature formulas provides the collective compactness of the approximation on the smooth parts of Γ. But on corners and edges the method still fails if $p \to q$. There we choose a special quadrature formula, namely, the rectangular rule. For this purpose let us denote by $\gamma \subset \Gamma$ *the set of all edge and corner points.* Let $d > 0$ be a fixed number *we choose beforehand.* d measures the neighborhood of γ and q where we use the rectangular rule. This neighborhood is chosen as

$$\Sigma := \{p \in \Gamma \mid |p - \gamma| < d\}$$

and

$$\Sigma_l := \Sigma \cap \{p \mid |p - p_l| < d\}. \tag{3.7}$$

Then the integral equation (3.2) is replaced by the following system of linear equations.[2]

[2] This discretization is a relatively crude approximation of the integral operator under investigation. It would be interesting to use Atkinson's factorization technique for this special kernel (see [4, 3.3] and the references given there).

$$
\begin{aligned}
u(p_l) = \frac{1}{2} u(p_l) - \frac{1}{2} \Bigg\{ & u(p_l) \frac{1}{2\pi} (2\pi - \Omega(p_l)) + \sum_{\substack{0<|p_j-p_l|\leqslant\delta \\ p_j\in\Sigma_l}} u(p_j) \frac{1}{2\pi} \int_{F_j} d\Omega_{p_l} \\
& + \sum_{\substack{0<|p_j-p_l|\leqslant\delta \\ p_j\notin\Sigma_l}} u(p_j) \frac{1}{2\pi} \frac{\mathbf{n}(p_j)\cdot(\mathbf{p}_j-\mathbf{p}_l)}{|p_j-p_l|^3} g_j \\
& - u(p_l) \frac{1}{2\pi} \Bigg[-\Omega(p_l) + \sum_{\substack{p_j\in\Sigma_l \\ j\neq l}} \int_{F_j} d\Omega_{p_l} + \sum_{\substack{p_j\notin\Sigma_l \\ j\neq l}} \frac{\mathbf{n}(p_j)\cdot(\mathbf{p}_j-\mathbf{p}_l)}{|p_j-p_l|^3} g_j \Bigg] \Bigg\} \\
& - \frac{1}{2} \Bigg\{ \frac{1}{2\pi} \sum_{j\neq l} u(p_j) [e^{ik|p_j-p_l|} - 1 - ik\,|p_j-p_l|\, e^{ik|p_j-p_l|}] \\
& \times \frac{\mathbf{n}(p_j)\cdot(\mathbf{p}_j-\mathbf{p}_l)}{|p_j-p_l|^3} g_j + \sum_{\substack{|p_j-p_l|>\delta \\ p_j\in\Sigma_l}} u(p_j) \frac{1}{2\pi} \int_{F_j} d\Omega_{p_l} \\
& + \sum_{\substack{|p_j-p_l|>\delta \\ p_j\notin\Sigma_l}} u(p_j) \frac{1}{2\pi} \frac{\mathbf{n}(p_j)\cdot(\mathbf{p}_j-\mathbf{p}_l)}{|p_j-p_l|^3} g_j \Bigg\} + f(p_l).
\end{aligned} \tag{3.8}
$$

The first curly bracketed term on the right-hand side of (3.8) represents a discretized form $Q_h u$ of Qu and the second term in curly brackets is the discretization $\Gamma_h u$ of Γu. The last, apparently additional, term in the discretized Qu is included (even though it vanishes as $h \to 0$) in order to obtain the discretized version of (1.25), namely

$$
\begin{aligned}
u_h(p_l) = & -\frac{1}{4\pi} \sum_{j\neq l} \alpha_{jl} [u_h(p_j) - u_h(p_l)] + f(p_l) \\
& - \frac{1}{4\pi} \sum_{j\neq l} u_h(p_j) \{e^{ik|p_j-p_l|} - 1 - ik\,|p_j-p_l|\, e^{ik|p_j-p_l|}\} g_j \\
& \times \frac{\mathbf{n}(p_j)\cdot(\mathbf{p}_j-\mathbf{p}_l)}{|p_j-p_l|^3},
\end{aligned} \tag{3.9}
$$

where the coefficients α_{jl} are

$$
\begin{aligned}
\alpha_{jl} &= \int_{F_j} d\Omega_{p_l} && \text{for } p_j \in \Sigma_l, \\
&= \frac{\mathbf{n}(p_j)\cdot(\mathbf{p}_j-\mathbf{p}_l)}{|p_j-p_l|^3} g_j && \text{for } p_j \notin \Sigma_l.
\end{aligned} \tag{3.10}
$$

These linear equations can be interpreted as an approximating functional equation in the space of boundary functions itself. A convenient way for formulating this concept is the introduction of *piecewise constant functions* which are constant on the pieces $F_j \subset \Gamma$. Since they are not included in $C^0(\Gamma)$ we extend C^0 to the space $B^1(\Gamma)$ of all functions of the first Baire's class furnished with the supremum norm $\|\cdot\|_0$. Then $C^0(\Gamma)$ becomes a closed subspace of B^1. We introduce the projection of B^1 to piecewise step functions,

$$P_h f(p) \equiv \lim_{r\to 0} \int_{|q-p_j|<r} f(q)\, dS_q \left(\int_{|q-p_j|<r} dS_q \right)^{-1} \qquad \text{for all } p \in F_j . \tag{3.11}$$

This is an extension of

$$P_h f(p) = f(p_k), \qquad p \in F_k , \qquad \text{for } f \in C^0(\Gamma). \tag{3.12}$$

With this definition, the linear equations (3.9) can also be interpreted as equations in $B^1(\Gamma)$,

$$u_h = L_{kh} u_h + f_h = \tfrac{1}{2}(I - K_{kh})\, u_h + f_h = \tfrac{1}{2}(I - Q_h - V_h)\, u_h + f_h , \tag{3.13}$$

where K_{kh} and L_{kh} are suitably defined by (3.9). In the following we shall show that these equations can be solved if h and k are small enough and that the solution can be found by the successive approximation

$$\begin{aligned} u_h^{(n+1)} &\equiv L_{kh} u_h^{(n)} + f_h , \qquad u^{(0)} \equiv f_h = P_h f, \\ &= \tfrac{1}{2}(I - Q_h - V_h)\, u_h^{(n)} + f_h . \end{aligned} \tag{3.14}$$

Using (3.12) for step functions, it is also clear that (3.14) is an abbreviation of the equations

$$\begin{aligned} u_h^{(n+1)}(p_l) \equiv -\frac{1}{4\pi} \sum_{j\neq l} \Big(& \alpha_{jl}(u_h^{(n)}(p_j) - u_h^{(n)}(p_l)) \\ & - u_h^{(n)}(p_j)\, \{e^{ik|p_j-p_l|} - 1 - ik\,|p_j - p_l|\, e^{ik|p_j-p_l|}\}\, g_j \\ & \frac{\mathbf{n}(p_j)\cdot(\mathbf{p}_j - \mathbf{p}_l)}{|p_j - p_l|^3} \Big) + f(p_l). \end{aligned} \tag{3.15}$$

However, it is more convenient for our purposes to use the decomposition of L_{kh} indicated in (3.14). Thus

$$\begin{aligned} Q_h u(p) \equiv & \Big[1 - (1/2\pi) \sum_{j\neq l} \alpha_{jl}\Big]\, P_h u(p_l) \\ & + (1/2\pi) \sum_{0<|p_j-p_l|<\delta} \alpha_{jl} P_h u(p_j), \qquad \text{for all } p \in F_l . \end{aligned} \tag{3.16}$$

The operator V_h is defined through (3.9)–(3.16).

Now, for the operator Q_h, the assumption (1.7) on the corners and edges together with $\| P_h \|_0 = 1$ implies the following properties:

LEMMA 3.1. *There exists a constant $\delta > 0$ depending only on Γ and d which determines a fineness $h_0(\delta) > 0$ such that for all $h \leqslant h_0(\delta)$ the operators Q_h are contractions and $I - \lambda Q_h$ are inverse stable:*

$$\| Q_h \|_0 \leqslant q_0 < 1 \qquad \textit{for all } h, \quad 0 \leqslant h \leqslant h_0(\delta), \tag{3.17}$$

$$\begin{aligned} \|(I - \lambda Q_h)^{-1} \|_0 \leqslant (1 - | \lambda | q_0)^{-1} \\ \textit{for } | \lambda | < (1/q_0) \quad \textit{where} \quad q_0 = \tfrac{1}{2}(1 + \omega) < 1. \end{aligned} \tag{3.18}$$

Proof. Recall that $\Gamma = \bigcup_{i=1}^{N} S_i$ where each S_i is a segment of a Lyapunoff surface and $\Sigma = \{p \mid p \in \Gamma, | p - \gamma | < d\}$ where γ is the set of corner and edge points. Define

$$d_{ij} \equiv \inf_{p \in S_i - \Sigma, q \in S_j - \Sigma} | p - q |$$

and

$$\tilde{d} \equiv \min_{i \neq j} d_{ij} .$$

Furthermore, let d_i denote the radius of the Lyapunoff sphere associated with S_i. Choose

$$2\delta < \min(\tilde{d}, d_1, ..., d_N).$$

Let $u \in B^1$ be an arbitrary function with $\| u \|_0 \leqslant 1$ and let us distinguish two cases according to the decomposition of L_k in (3.2) and in (3.16).

(a) $| p_l - \gamma | \geqslant d \geqslant \delta$. In this case, all the pieces F_j belonging to $| p_j - p_l | \leqslant \delta$ lie on a *common* Lyapunoff surface S_i. Hence, the kernel in $d\Omega_q$ is weakly singular [21, I, Sect. 1]. Using the representation (1.5) and the abbreviation

$$d\Omega_q(p) = \phi(p, q) \mid p - q \mid^{-3} dS_p , \tag{3.19}$$

where

$$\phi(p, q) = \mathbf{n}(p) \cdot (\mathbf{p} - \mathbf{q}), \tag{3.20}$$

a standard treatment, e.g., [21], guarantees that there are real positive constants $\alpha \leqslant 1$ and C such that

$$\begin{aligned} | \phi(p, q)| &\leqslant C \mid p - q \mid^{1+\alpha}, \\ | \phi(p, q) - \phi(p', q)| &\leqslant C(\mid p - p' \mid^{1+\alpha} + \mid p - q \mid \mid p - p' \mid^{\alpha}), \end{aligned} \tag{3.21}$$

when p, p', and q lie on a common Lyapunoff surface within the Lyapunoff sphere. The first inequality follows from [21, I, (17)] and the second follows from the first, the Lyapunoff assumption on the angle between two normals [21, I, (1)], and the fact that

$$\begin{aligned}
&|\phi(p, q) - \phi(p', q)| \\
&\quad = |(\mathbf{n}(p) - \mathbf{n}(p')) \cdot (\mathbf{p} - \mathbf{q}) - \mathbf{n}(p') \cdot (\mathbf{p}' - \mathbf{q}) + \mathbf{n}(p') \cdot (\mathbf{p} - \mathbf{q})| \\
&\quad \leqslant |\mathbf{n}(p) - \mathbf{n}(p')| \, |\mathbf{p} - \mathbf{q}| + |\phi(p', p)| \\
&\quad \leqslant (2 - 2\cos(\mathbf{n}(p), \mathbf{n}(p')))^{1/2} \, |p - q| + |\phi(p', p)| \\
&\quad \leqslant 2 \, |\sin \tfrac{1}{2}(\mathbf{n}(p), \mathbf{n}(p'))| \, |p - q| + |\phi(p', p)| .
\end{aligned}$$

Because p_l is neither a corner nor an edge point, the special relation $\Omega(p_l) = 2\pi$ holds and $\Sigma_l = \varnothing$.

Now (3.16) can be estimated with $|u| \leqslant 1$ as

$$|Q_h u(p_l)| \leqslant \frac{1}{2\pi} \sum_{\substack{j=l \\ |p_j - p_l| \leqslant \delta}} g_j \frac{|\phi(p_j, p_l)|}{|p_j - p_l|^3} + \frac{1}{2\pi} \left| \sum_{j \neq l} g_j \frac{\phi(p_j, p_l)}{|p_j - p_l|^3} - 2\pi \right| .$$

Decomposing the second sum into one on $\bigcup_{0 < |p_j - p_l| \leqslant \delta} F_j$ and another on the rest of Γ, introducing integrals using

$$1 = (1/|F_j|) \int_{F_j} dS \qquad \text{and} \qquad 2\pi = \sum_{j=1}^{M} \int_{F_j} d\Omega_{p_l}(p), \qquad (p_l \notin \gamma),$$

and adding and subtracting $\phi(p, p_l)/|p - p_l|^3$ appropriately, the above can be written as

$$\begin{aligned}
|Q_h u(p_l)| \leqslant\ & \frac{1}{2\pi} \left| \int_{F_l} d\Omega_{p_l} \right| \\
& + \frac{1}{2\pi} \sum_{\substack{j \neq l \\ |p_j - p_l| \leqslant \delta}} \frac{g_j}{|F_j|} \int_{F_j} \left| \frac{\phi(p_j, p_l)}{|p_j - p_l|^3} - \frac{\phi(p, p_l)}{|p - p_l|^3} \right| dS \\
& + \frac{1}{2\pi} \sum_{\substack{j = l \\ |p_j - p_l| \leqslant \delta}} \frac{g_j}{|F_j|} \int_{F_j} \frac{|\phi(p, p_l)|}{|p - p_l|^3} dS \\
& + \frac{1}{2\pi} \sum_{\substack{|p_j - p_l| \leqslant \delta \\ j \neq l}} \int_{F_j} |d\Omega_{p_l}| \qquad\qquad (3.22) \\
& + \frac{1}{2\pi} \sum_{\substack{j = l \\ |p_j - p_l| \leqslant \delta}} \frac{g_j}{|F_j|} \int_{F_j} \left| \frac{\phi(p_j, p_l)}{|p_j - p_l|^3} - \frac{\phi(p, p_l)}{|p - p_l|^3} \right| dS
\end{aligned}$$

$$+\frac{1}{2\pi}\sum_{\substack{j\neq l\\ |p_j-p_l|\leqslant\delta}}\frac{g_j}{|F_j|}\int_{F_j}\frac{|\phi(p,p_l)|}{|p-p_l|^3}\,dS$$

$$+\frac{1}{2\pi}\left|\sum_{|p_j-p_l|\leqslant\delta}\left(g_j\frac{\phi(p_j,p_l)}{|p_j-p_l|^3}-\int_{F_j}d\Omega_{p_l}\right)\right|.$$

Recall that (in case (a)) all the F_j for $|p_j-p_l|\leqslant\delta$ are parts of a common Lyapunoff surface S_l. Choosing $h_0<\delta$, it follows that for $h\leqslant h_0$,

$$\bigcup_{|p_j-p_l|\leqslant\delta}F_j\subset S_l\cap\{p\mid |p-p_l|\leqslant 2\delta\}.$$

This expresses the fact that while some points $p\in F_j$ may be further from p_l than δ, they can't be more than 2δ away. Hence

$$\frac{1}{2\pi}\sum_{\substack{|p_j-p_l|\leqslant\delta\\ j\neq l}}\int_{F_j}|d\Omega_{p_l}(p)| \tag{3.23}$$
$$\leqslant\frac{1}{2\pi}\int_{S_l\cap\{p\mid|p-p_l|\leqslant 2\delta\}}|d\Omega_{p_l}(p)|=\frac{1}{2\pi}\int_{S_l\cap\{p\mid|p-p_l|\leqslant 2\delta\}}\frac{|\phi(p,p_l)|}{|p-p_l|^3}\,dS.$$

Furthermore the way in which δ was chosen ensures that the region $S_l\cap\{p\mid|p-p_l|\leqslant 2\delta\}$ lies within the Lyapunoff sphere; hence (3.21) may be used to obtain the estimate

$$\frac{1}{2\pi}\sum_{\substack{|p_j-p_l|\leqslant\delta\\ j\neq l}}\int_{F_j}|d\Omega_{p_l}(p)|\leqslant c_1\int_{S_l\cap\{p\mid|p-p_l|\leqslant 2\delta\}}|p-p_l|^{\alpha-2}\,dS, \tag{3.24}$$
$$\leqslant c_2\delta^\alpha$$

where c_1 and c_2 are constants depending only on Γ [21, I, (38)].

For the second kind of terms in (3.22) we observe that for $p, p_j\in F_j$ and $p_l\notin F_j$, there is a constant c_3 independent of h such that

$$1/|p_j-p_l|\leqslant c_3/|p-p_l|. \tag{3.25}$$

If $|p_j-p_l|\geqslant|p-p_l|$ then (3.25) follows immediately, whereas if $|p_j-p_l|\leqslant|p-p_l|$ the result follows from the triangle inequality

$$|p-p_l|\leqslant|p-p_j|+|p_j-p_l|$$

and the fact that with Eq. (3.6) and $p, p_j\in F_j$,

$$|p-p_j|\leqslant h\leqslant c\,|p_j-p_l|.$$

Using (3.6), (3.21) and (3.25) we get

$$\begin{aligned}
&\frac{1}{2\pi} \sum_{\substack{j=l \\ |p_j-p_l|\leqslant\delta}} \frac{g_j}{|F_j|} \int_{F_j} \left| \frac{\phi(p_j, p_l)}{|p_j-p_l|^3} - \frac{\phi(p, p_l)}{|p-p_l|^3} \right| dS \\
&\leqslant c' \sum_{\substack{j\neq l \\ |p_j-p_l|\leqslant\delta}} \int_{F_j} \left(\frac{1}{|p_j-p_l|^{2-\alpha}} + \frac{1}{|p-p_l|^{2-\alpha}} \right) dS \qquad (3.26) \\
&\leqslant c'' \sum_{\substack{j=l \\ |p_j-p_l|\leqslant\delta}} \int_{F_j} |p-p_l|^{\alpha-2}\, dS,
\end{aligned}$$

in which form (3.24) applies to give a similar bound,

$$\frac{1}{2\pi} \sum_{\substack{|p_j-p_l|\leqslant\delta \\ j\neq l}} \frac{g_j}{|F_j|} \int_{F_j} \left| \frac{\phi(p_j, p_l)}{|p_j-p_l|^3} - \frac{\phi(p, p_l)}{|p-p_l|^3} \right| dS \leqslant c''\delta^\alpha. \qquad (3.27)$$

Using (3.24) and (3.27) in (3.22) leads to

$$\begin{aligned}
&|Q_h u(p_l)| \\
&\leqslant \frac{1}{2\pi} \int_{F_l} |d\Omega_{p_l}(p)| + c''\delta^\alpha + \frac{1}{2\pi} \left| \sum_{|p_j-p_l|>\delta} \left(g_j \frac{\phi(p_j, p)}{|p_j-p_l|^3} - \int_{F_j} d\Omega_{p_l} \right) \right|.
\end{aligned}$$

Since p_l is not a corner or an edge point and $F_l \cap \gamma = \varnothing$, (3.19), (3.21) imply

$$(1/2\pi) \int_{F_l} |d\Omega_{p_l}| \leqslant \tilde{c}' \int_{F_l} |p-p_l|^{\alpha-2}\, dS \leqslant \tilde{c}''\delta^\alpha, \qquad (3.29)$$

where the constant $\tilde{c}''$ is independent of p and δ. Hence there is a $\delta > 0$ such that

$$|Q_h u(p_l)| \leqslant \frac{1}{4}(1+\omega) + \frac{1}{2\pi} \left| \sum_{|p_j-p_l|>\delta} \left(g_j \frac{\phi(p_j, p_l)}{|p_j-p_l|^3} - \int_{F_j} d\Omega_{p_l} \right) \right|. \qquad (3.30)$$

With $\delta > 0$ chosen and fixed, the last term represents the absolute value of the difference of the quadrature formula approximation of the integral operator operating on the special function identically 1. For $|p_j - p_l| > \delta$ the kernel $d\Omega_{p_l}(p)$ is continuous in both variables. Hence, the last term converges uniformly to zero if $h \to 0$ according to [4, Proposition 2.1]. That implies the existence of some $h_0 > 0$ such that

$$|Q_h u(p_l)| \leqslant \tfrac{1}{2}(1+\omega) \qquad \text{for all } h \leqslant h_0 \quad \text{and} \quad |u| \leqslant 1.$$

This is (3.17) in case (a).

(b) $|p_l - \gamma| < d$. In this case we have to take into account the different definitions of the α_{jl} for $p_j \in \Sigma_l$ and $p_j \notin \Sigma_l$. Now, (3.16) implies

$$\begin{aligned}|Q_h u(p_l)| \leqslant{}& \frac{1}{2\pi} \sum_{\substack{0<|p_j-p_l|\leqslant\delta\\ p_j\in\Sigma_l}} \left|\int_{F_j} d\Omega_{p_l}\right| \\ &+ \frac{1}{2\pi} \sum_{\substack{0<|p_j-p_l|\leqslant\delta\\ p_j\notin\Sigma_l}} g_j \frac{|\phi(p_j, p_l)|}{|p_j - p_l|^3} + \frac{1}{2\pi} |2\pi - \Omega(p_l)| \\ &+ \frac{1}{2\pi} \left| \sum_{\substack{p_j\in\Sigma_l\\ j\neq l}} \int_{F_j} d\Omega_{p_l} + \sum_{\substack{p_j\notin\Sigma_l\\ j\neq l}} g_j \frac{\phi(p_j, p_l)}{|p_j - p_l|^3} - \Omega(p_l) \right|.\end{aligned}$$

Recalling the definition of W_δ in (1.8) and using the facts that $h \leqslant h_0 < \delta$ and

$$\Omega(p_l) = \sum_{p_j\in\Sigma_l} \int_{F_j} d\Omega_{p_l} + \sum_{p_j\notin\Sigma_l} \int_{F_j} \frac{\phi(p, p_l)}{|p - p_l|^3} dS,$$

we obtain

$$\begin{aligned}|Q_h u(p_l)| \leqslant{}& W_{2\delta}(p_l) + \frac{1}{\pi} \sum_{\substack{0<|p_j-p_l|\leqslant\delta\\ p_j\notin\Sigma_l}} \frac{g_j}{|F_j|} \int_{F_j} \left| \frac{\phi(p_j, p_l)}{|p_j - p_l|^3} - \frac{\phi(p, p_l)}{|p - p_l|^3} \right| dS \\ &+ \frac{1}{\pi} \sum_{\substack{0<|p_j-p_l|\leqslant\delta\\ p_j\notin\Sigma_l}} \frac{g_j}{|F_j|} \int_{F_j} \frac{|\phi(p, p_l)|}{|p - p_l|^3} dS \\ &+ \frac{1}{2\pi} \left| \sum_{\substack{|p_j-p_l|>\delta\\ p_j\notin\Sigma_l}} \left(g_j \frac{\phi(p_j, p_l)}{|p_j - p_l|^3} - \int_{F_j} d\Omega_{p_l} \right) \right|. \qquad (3.31)\end{aligned}$$

Because Γ is a piecewise Lyapunoff surface without self-intersections, for the remaining terms with $p_j \notin \Sigma_l$ only two different cases are possible: Either

(i) p_j and p_l are lying on a *common* Lyapunoff surface S_i; or

(ii) p_j and p_l are on *different* surface pieces $S_i \neq S_{i'}$.

When $p_l \in \Sigma$ and $p_j \notin \Sigma_l$ then $|p_j - p_l| > d$ holds according to (3.7). When $p_l \notin \Sigma$ then $|p_j - \gamma| \geqslant d$. Because every path on Γ from $p_j \in S_{i'}$ to $p_l \in S_i$ intersects γ, and Γ has no self-intersections, the distance $|p_j - p_l|$ can be estimated below by $c'|p_j - \gamma|$ with $c' > 0$, $c' \leqslant 1$ depending only on Γ. Hence the inequality

$$|p_j - p_l| \geqslant c'd, \qquad c' > 0 \qquad (3.32)$$

holds.

For all points p_j arising in case (i), the terms can be estimated as in (a). For all points arising in case (ii), if $\delta > 0$ is chosen small enough ($\delta < c'd$), the second and third parts in (3.31) do not appear, whereas the kernel $d\Omega_p$ is continuous in both variables. Hence, as in case (a) we get from (3.31)

$$|Q_h u(p_l)| \leq \| V_{2\delta}(p_l) + \frac{1}{2\pi} \Big| \sum_{\substack{|p_j - p_l| < \delta \\ p_j \notin \Sigma_l}} \Big(g_j \frac{\phi(p_j, p_l)}{|p_j - p_l|^3} - \int_{F_j} d\Omega_{p_l}\Big) \Big|$$

$$\leq \frac{1 + 2\omega}{3} + \frac{1}{2\pi} \Big| \sum_{\substack{|p_j - p_l| < \delta \\ p_j \notin \Sigma_l}} \Big(g_j \frac{\phi(p_j, p_l)}{|p_j - p_l|^3} - \int_{F_j} \frac{\phi(p, p_l)}{|p - p_l|^3} dS\Big) \Big|$$

for some $\delta > 0$ small enough. As before, for fixed δ and $h \to 0$ the second term tends uniformly to zero, hence is less than $(1 - \omega)/6$ for $h < h_0$ with h_0 small enough, and (3.17) holds again.

Thus, (3.17) is proven. The second part, (3.18), follows from (3.17) by a well-known theorem (e.g., [4, p. 2]), hence Lemma 3.1 is proven.

For the operators $V_h = K_{l,h} - Q_h$ and a fixed $\delta > 0$, the kernel of the corresponding integral operator, viz. [cf. (3.4)],

$$\begin{aligned} &\frac{1}{2\pi} \cdot \begin{cases} \dfrac{\mathbf{n}(p) \cdot (\mathbf{p} - \mathbf{q})}{|p - q|^3} & \text{for } |p - q| \geq \delta \\ 0 & \text{for } |p - q| < \delta \end{cases} \\ &+ \frac{1}{2\pi} \{e^{ik|p-q|}(1 - ik|p - q|) - 1\} \frac{\mathbf{n}(p) \cdot (\mathbf{p} - \mathbf{q})}{|p - q|^3} \end{aligned} \tag{3.33}$$

is continuous in either domain of definition. Then it follows from Anselone's approach [4, Chap. 2, Theorem 2.13] that

$$\begin{aligned} &\text{(a)} \quad \{V_h\} \textit{ is a collectively compact family;} \\ &\text{(b)} \quad V_h \to V \quad \textit{as} \quad h \to 0; \\ &\text{(c)} \quad \| V_h \|_0 \to \| V \|_0 \quad \textit{as} \quad h \to 0; \\ &\text{(d)} \quad V \textit{ is compact.} \end{aligned} \tag{3.34}$$

These properties together with the inverse stability (3.18) imply the following lemma.

LEMMA 3.2. *The family*

$$\{[I - \lambda(I - \lambda Q)^{-1} V] - [I - \lambda(I - \lambda Q_h)^{-1} V_h]\} \tag{3.35}$$

is collectively compact for each $|\lambda| < 1/q_0$.

Proof. The operators in (3.35) can be written as

$$\{\cdots\} = \lambda((I - \lambda Q_h)^{-1} - (I - \lambda Q)^{-1})\, \Gamma + \lambda(I - \lambda Q_h)^{-1}(\Gamma_h - \Gamma). \tag{3.36}$$

The inverse stability (3.18) guarantees

$$(I - \lambda Q_h)^{-1} \to (I - \lambda Q)^{-1}. \tag{3.37}$$

Hence, the first term in (3.36) is collectively compact [4, Proposition 1.8]. The second term is collectively compact because of (3.18) and (3.34) [4, Propositions 4.2, 2].

According to Theorem 2.2, we choose a constant ρ,

$$1 < \rho < \min\{2/(1 + q_0), \mu_1\}, \tag{3.38}$$

defining a disk M (2.7) in the μ-plane. The inverse image of M with respect to (2.5) then is (see Fig. 1)

$$M' = \{\lambda \mid 2\lambda/(\lambda - 1) \in M\} \tag{3.39}$$

and the estimate

$$|\lambda| \leqslant \rho/(2 - \rho) < 1/q_0 \tag{3.40}$$

is easily established. This enables us to prove the following Theorem.

THEOREM 3.1. *For any ρ satisfying* (3.38) *there exist constants $k_\rho > 0$ and $h_1 > 0$ such that for all k with $|k| \leqslant k_\rho$, for all $h \leqslant h_1$ and for all $\mu \in M$ the operators $I - \mu L_{kh}$ are uniformly inverse stable; i.e., there exists a constant c, such that*

$$\|(I - \mu L_{kh})^{-1}\|_0 \leqslant c, \qquad \textit{for } \mu \in M, \quad |k| \leqslant k_\rho, \quad h \leqslant h_1. \tag{3.41}$$

The proof is similar to that of [8, Satz 4]; see also [4, Theorem 4.16].

Proof. From Theorem 2.2 it follows that M belongs to the resolvent set of $(I - \mu L_k)$ for all k, $|k| \leqslant k_\rho$ with k_ρ chosen there. Consequently, for any $\lambda' \in M'$ the operator

$$I - \lambda' K_k = (I - \lambda' Q)\,[I - \lambda'(I - \lambda' Q)^{-1}\, \Gamma]$$

has an inverse, as does $[I - \lambda'(I - \lambda' Q)^{-1}\, \Gamma]$. Hence, for any $\lambda' \in M'$ and any k' with $|k'| \leqslant k_\rho$ there exist constants c_0 and h_0 such that

$$\|[I - \lambda'(I - \lambda' Q_h)^{-1}\, \Gamma_h]^{-1}\|_0 \leqslant c_0 \tag{3.42}$$

for all $h \leqslant h_0$ [8, Hilfssatz 5b; 4, Theorem 1.11]. Continuity in λ' and k' ensures that there exists a $\delta > 0$ such that

$$\|[I - \lambda(I - \lambda Q_h)^{-1} V_h]^{-1}\|_0 \leqslant 2c_0$$

for all $h \leqslant h_0$ and $|\lambda - \lambda'| + |k - k'| < \delta$. Heine–Borel's theorem with respect to the compact set $M' \times \{k \mid |k| \leqslant k_\rho\}$ implies that there exist constants $h_1 > 0$ and $\tilde{c}_0 > 0$ such that the estimate

$$\|[I - \lambda(I - \lambda Q_h)^{-1} V_h]^{-1}\|_0 \leqslant \tilde{c}_0 \tag{3.43}$$

holds for all $h \leqslant h_1$, all $\lambda \in M'$ and all k, $|k| \leqslant k_\rho$.

With (3.14)–(3.16) relating L_{kh}, Q_h and V_h and since $\mu = 2\lambda/(\lambda - 1)$, it follows that

$$[I - \mu L_{kh}]^{-1} = (1 - (\mu/2))^{-1} [I - \lambda(I - \lambda Q_h)^{-1} V_h]^{-1} (I - \lambda Q_h)^{-1},$$

from which (3.18) and (3.43) imply

$$\|(I - \mu L_{kh})^{-1}\|_0 \leqslant |(1 - (\mu/2))^{-1}| \, \tilde{c}_0 \, |(1 - |\lambda| q_0)^{-1}| . \tag{3.44}$$

Since (cf. (2.7))

$$|\mu| \leqslant \rho < 2,$$

it follows that

$$|(1 - (\mu/2))^{-1}| \leqslant (1 - (\rho/2))^{-1}. \tag{3.45}$$

Similarly, (3.40) implies that

$$(1 - |\lambda| q_0)^{-1} \leqslant (1 - (\rho/(2 - \rho)) q_0)^{-1}. \tag{3.46}$$

Incorporating (3.45) and (3.46) in (3.44) establishes that

$$\|[I - \mu L_{kh}]^{-1}\|_0 \leqslant (1 - (\rho/2))^{-1} \tilde{c}_0(1 - (\rho/(2 - \rho)) q_0)^{-1}, \tag{3.47}$$

for all $\mu \in M$, $|k| \leqslant k_\rho$ and $h \leqslant h_1$, which is the desired inequality (3.41).

The uniform inverse stability of $(I - \mu L_{kh})$ which is formulated in Theorem 3.1 leads to the announced result on the convergence of the Neumann series

$$u_h^{(n)} = \sum_{j=0}^{n} L_{kh}^j f_h = L_{kh} u_h^{(n-1)} + f_h , \qquad u_h^{(0)} := f_h . \tag{3.48}$$

To see this we use (3.41) in Cauchy's integral formula [50, p. 261]:

$$L_{kh}^{j} = (1/2\pi i) \int_{|\mu|=\rho} (1/\mu^{j+1}) (I - \mu L_{kh})^{-1} \, d\mu, \tag{3.49}$$

$$\| L_{kh}^{j} \|_0 \leqslant c_1 \rho^{-j}, \tag{3.50}$$

where $\rho > 1$ (3.38). Consequently, the geometric series

$$\sum_{j=0}^{\infty} c_1 \rho^{-j} \| f_h \|_0$$

majorizes (3.48). Thus, one can conclude the

COROLLARY. *The Neumann series* (3.48) *converges and the following error estimate holds:*

$$\| u_h - u_h^{(n)} \|_0 \leqslant c_1 \| f_h \|_0 (1/\rho^n) \cdot (1/(\rho - 1)) \tag{3.51}$$

uniformly for all k, $|k| \leqslant k_\rho$ *and* $h \leqslant h_1$.

APPENDIX A: ON THE CONTRACTION OF THE OPERATOR L_k (1.25) FOR A CONVEX SCATTERER

Let Γ be the boundary of a *convex* scattering body so that

$$\Omega(q) = \int_{\Gamma - \{q\}} d\Omega_q(p) = \int_{\Gamma - \{q\}} | d\Omega_q(p)| \leqslant 2\pi.$$

For such regions C. Neumann established the inequality

$$(1/4\pi) \int_{\Gamma - \{p_1\} - \{p_2\}} | d\Omega_{p_1}(p) - d\Omega_{p_2}(p)| \leqslant \delta < 1 \quad \text{for all } p_1, p_2 \in \Gamma, p_1 \neq p_2, \tag{A1}$$

where δ is a suitable constant (see, e.g., [19, pp. 203ff]).

The original proof dealt with curves in R^2 rather than surfaces in R^3 but the same arguments yield the result (A1).

Schober, however, pointed out that the classical proof contains a gap in the case when there are corner points and provided a valid proof for curves Γ with corner points but without spines [52]. His proof remains valid for three-dimensional convex bodies with smooth enough boundaries; however, it is not clear whether the general surfaces considered in the preceding chapters are "smooth enough." *Therefore, let us assume that* (A1) *is satisfied for the scattering body under investigation.* Let α be any constant with

$$0 < \alpha < 1 - \delta. \tag{A2}$$

With α fixed, the expression

$$| u | \equiv \sup_{p_1, p_2 \in \Gamma} | u(p_1) - u(p_2)| + \alpha \sup | u | \tag{A3}$$

defines a norm on $C^0(\Gamma)$ which is equivalent to the norm $\| \cdot \|_0$. With respect to (A3), L_k (1.25) becomes a contraction; more precisely, the following lemma holds.

LEMMA A. *For all k small enough, i.e., satisfying*

$$(| k |^2/2\pi) \max\{1, e^{-\operatorname{Im} k D}\} \int_\Gamma dS \leq \alpha(1 + \delta + \alpha)/(4 + 2\alpha), \tag{A4}$$

the estimate

$$| L_k u | \leq \rho | u | \tag{A5}$$

holds for all $u \in C^0(\Gamma)$ with

$$\rho = \tfrac{1}{2}(1 + \delta + \alpha) < 1$$

where D is the diameter of Γ and δ is the bound appearing in (A1).

Proof. From (1.9) and (1.25) follows the identity

$$\begin{aligned} &L_k u(p_1) - L_k u(p_2) \\ &\quad = u(p_1)\,(\Omega(p_1)/4\pi) - u(p_2)\,(\Omega(p_2)/4\pi) - (1/4\pi) \int_\Gamma u(p)\,(d\Omega_{p_1}(p) - d\Omega_{p_2}(p)) \\ &\qquad + (L_k - L_0)\,u(p_1) - (L_k - L_0)\,u(p_2). \end{aligned} \tag{A6}$$

For an arbitrary point $q \in \Gamma$ this can be written as

$$\begin{aligned} &L_k u(p_1) - L_k u(p_2) \\ &\quad = u(p_1)\,(\Omega(p_1)/4\pi) - u(p_2)\,(\Omega(p_2)/4\pi) + u(q)\,(1/4\pi)\,[\Omega(p_2) - \Omega(p_1)] \\ &\qquad - (1/4\pi) \int_\Gamma (u(p) - u(q))\,(d\Omega_{p_1}(p) - d\Omega_{p_2}(p)) \\ &\qquad + (L_k - L_0)\,u(p_1) - (L_k - L_0)\,u(p_2). \end{aligned}$$

Let $q_0(q_1)$ be minimal (maximal) points of u,

$$u(q_0) = \min_{p \in \Gamma} u(p), \qquad u(q_1) = \max_{p \in \Gamma} u(p).$$

Because u is continuous, there exists $q \in \Gamma$ such that

$$u(q) = \tfrac{1}{2}(u(q_0) + u(q_1)).$$

Choose q in (A7) to be this point. It follows that for any $p \in \Gamma$,

$$|u(p) - u(q)| = \tfrac{1}{2}\,|(u(q_1) - u(p)) - (u(p) - u(q_0))|\,.$$

The right-hand side is the absolute value of the difference of two *nonnegative* numbers. Hence,

$$|u(p) - u(q)| \leqslant \tfrac{1}{2}\max\{|u(q_1) - u(p)|\,,\,|u(p) - u(q_0)|\},$$

and, therefore,

$$|u(p) - u(q)| \leqslant \tfrac{1}{2}\operatorname{osc} u. \tag{A8}$$

From (A7) it follows that

$$\begin{aligned} &|L_k u(p_1) - L_k u(p_2)| \\ &\quad\leqslant (1/4\pi)\,|u(p_1) - u(q)|\,\Omega(p_1) + (1/4\pi)\,|u(p_2) - u(q)|\,\Omega(p_2) \\ &\quad\quad + (1/4\pi)\int_{\Gamma-\{p_1\}-\{p_2\}} |u(p) - u(q)|\,|\,d\Omega_{p_2}(p) - d\Omega_{p_1}(p)| \\ &\quad\quad + 2\,\|L_k - L_0\|\,\|u\|_0\,. \end{aligned} \tag{A9}$$

Using (A1), (A4), (A8), (1.27), and the fact that $\Omega(p_{\substack{1\\2}}) \leqslant 2\pi$, (A9) yields

$$\operatorname{osc} L_k u \leqslant ((1+\delta)/2)\operatorname{osc} u + 2\,\|u\|_0\,\alpha(1+\delta+\alpha)/(4+2\alpha). \tag{A10}$$

Further, with (1.25), (A4) and (A8)

$$\|L_k u\|_0 \leqslant \tfrac{1}{2}\operatorname{osc} u + \|u\|_0\,\alpha(1+\delta+\alpha)/(4+2\alpha). \tag{A11}$$

With the definition (A3) we obtain the desired inequality

$$\begin{aligned} |\, L_k u \,| &= \operatorname{osc} L_k u + \alpha\,\|L_k u\|_0\,, \\ &\leqslant ((1+\delta+\alpha)/2)\operatorname{osc} u + \alpha(1+\delta+\alpha)\,((2+\alpha)/(4+2\alpha))\,\|u\|_0\,, \\ &\leqslant ((1+\delta+\alpha)/2)\,[\operatorname{osc} u + \alpha\,\|u\|_0] = \rho\,|\, u\,|\,. \end{aligned}$$

APPENDIX B: THE BOUNDARY FLOW OF THE GENERATED SOLUTION (1.15)

Here we shall prove the following lemma.

LEMMA B. *Let $g \in C_v$ be given and let u be the solution of the integral equation* (1.24) *on the boundary surface Γ. Then the extension of u defined by* (1.15) *possesses a boundary flow $\partial u/\partial n \in C_v$ and $\partial u/\partial n = g$, provided that k is not a critical value.*

Proof. First we construct a solution w of the exterior problem (1.1), (1.2), (1.3) as a single layer potential

$$w = (1/2\pi) \int_\Gamma (e^{ikR}/R)\, dh, \tag{B1}$$

where $h \in C^{0*}$ is a suitable function of bounded total variation. Hence, as in the classical approach, the density h is to be found as the solution of the equations

$$g(F) = -h(F) - K_k{}^*h(F), \tag{B2}$$

$$g(F) = -h(F) - K_0{}^*h(F)$$
$$-\frac{1}{2\pi}\int_F \left\{\int_\Gamma \left(\frac{\mathbf{n}(p)\cdot(\mathbf{q}-\mathbf{p})}{R^3}\right)(e^{ikR} - 1 - ikRe^{ikR})\, dh(q)\right\} dS_p \tag{B3}$$

where $F \subset \Gamma$ denotes an arbitrary Borel measurable set on Γ and $K_0{}^*$ is the adjoint operator to K_0 operating on C^{0*}. With (2.2) it follows that u is a solution of (1.24) if and only if

$$2f = -u - K_k u. \tag{B4}$$

Since this equation has a unique solution in $C^0(\Gamma)$ and since (B2) is the equation conjugate to (B4), it follows that (B2) has a unique solution $h \in C^{0*}$. With this solution, the second term in (B3) has an integrand fulfilling

$$\left|\int_\Gamma \frac{\mathbf{n}(p)\cdot(\mathbf{q}-\mathbf{p})}{R^3}(e^{ikR} - 1 - ikRe^{ikR})\, dh(q)\right|$$
$$\leq \max_{p,q\in\Gamma}(1/R^2)\,|\, e^{ikR} - 1 - ikRe^{ikR}\,|\,\|\, h\,\|_{C^{0*}} < \infty. \tag{B5}$$

Hence this term defines a set function with L^∞ density belonging to C_v. This implies, together with $g \in C_v$, that

$$h + K_0{}^*h \in C_v. \tag{B6}$$

Theorem 5 in [11] yields $h \in C_v$. With this density h, w (B1) becomes the solution of (1.1), (1.2), (1.3) and ***has the boundary flow g.***

This can be shown by the use of Theorem 4 in [11] and an investigation similar to that of the limit (1.19). The property $h \in C_v$ implies $w \in C^0(R^3)$. Because w is a solution of (1.1), (1.2), (1.3) in the desired class, it can be represented by (1.15). Subtraction from the equation for u leads to the integral equation

$$w - u = L_k(u - w) \qquad \text{on } \Gamma \tag{B7}$$

for the continuous boundary values $w - u\,|_\Gamma$. Since k is not a critical value this implies $w = u$ on Γ and, through (1.15), also all over $R^3 - G$. Hence u has the desired boundary flow

$$\partial u/\partial n = \partial w/\partial n = g \in C_c\,.$$

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